The proof of our next theorem, Theorem 2.1, is similar to the proof of Theorem 2.0. However, the auxiliary game G_1^2 defined in Theorem 2.1 has some ordinal auxiliary moves λ_i^0 which the auxiliary game, G_0^2, of Theorem 2.0 does not have. In the following proof, we integrate a w.s. s_1^2 for G_1^2 with respect to the λ_i^0's using indiscernibles for $L[\#_2^1]$; otherwise, the integration of s_1^2 is analogous to the integration of s_0^2 (in Theorem 2.0).

Theorem 2.1. If $L[\#_2^1]$ has indiscernibles, then $\mathrm{Det}(\Pi_1^0, \Pi_1^0)_+^*$.

Proof: Assume $L[\#_2^1]$ has an uncountable set C_1^2 of indiscernibles. Let $B_2, B_1 \in \Pi_1^0$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and D strongly witness $\mathrm{A} \in (\Pi_1^0, \Pi_1^0)_+^*$. Wlog $B_2 \subseteq B_1$. Let $\langle D_\alpha | \alpha < \omega \cdot m \rangle$ witness that D is $\omega \cdot m - \Pi_1^0$, where $m \in \mathbf{N}$. Then there exist R_1 and R_2 in Δ_1^0 such that

i.) $B_i(x, n) \leftrightarrow \forall k R_i(\bar{x}(k), n)$,

and

ii.) if $\neg R_i(\bar{x}(k), n)$ and $\forall j < k \; R_i(\bar{x}(j), n)$, then k is odd.

We show that G_A has a w.s. s. Condition (ii) helps to simplify the proof.

We describe an open game G_1^2 which has a w.s. $s_1^2 \in L[\#_2^1]$. We integrate s_1^2 to get the w.s. $s \in L(\#_3^1(0))$ for G_A. G_1^2 is similar to the game G_0^2. The moves of G_1^2 and G_0^2 are the same with one exception: Once II plays $\langle \hat{u}_i, u_i \rangle = \langle 0, u_i \rangle$, in G_0^2 I and II respectively play integer moves $x(2i)$ and $x(2i+1)$, whereas, in G_1^2 I and II respectively play ordinals λ_{2i}^0 and λ_{2i+1}^0 with their respective integer moves $x(2i)$ and $x(2i+1)$. If II plays $\hat{u}_k = 0$ for all $k \in \omega$, then the play of G^2_1 is

$$\begin{array}{lcccccccc} \mathrm{I} & U_0 & & x(0),\lambda^0_0 & & U_1 & & x(2),\lambda^0_2 & & \cdots & U_{n-1} & & \\ \mathrm{II} & & \langle 0,u_0\rangle & & x(1),\lambda^0_1 & & \langle 0,u_1\rangle & & x(3),\lambda^0_3 & & & \langle 0,u_{n-1}\rangle & \cdots. \end{array}$$

If II plays $\hat{u}_k = 1$ and $\hat{t}_j = 0$ for all $j \in \omega$, then the play of G^2_1 is

$$\begin{array}{lcccccccc} \mathrm{I} & U_0 & & x(0),\lambda^0_0 & & U_1 & & x(2),\lambda^0_2 & \\ \mathrm{II} & & \langle 0,u_0\rangle & & x(1),\lambda^0_1 & & \langle 0,u_1\rangle & & x(3),\lambda^0_3 \cdots \end{array}$$

$$\begin{array}{cccccc} \cdots U_{k-1} & & x(2k-2),\lambda^0_{2k-2} & & U_k & \\ & \langle 0,u_{k-1}\rangle & & x(2k-1),\lambda^0_{2k-1} & & \langle 1,-\rangle \end{array}$$

$$\begin{array}{cccccccc} T_0 & & x(2k),\lambda_0 & & T_1 & & x(2k+2),\lambda_2 & \\ & \langle 0,t_0\rangle & & x(2k+1),\lambda_1 & & \langle 0,t_1\rangle & & x(2k+3),\lambda_3 \cdots. \end{array}$$

If $\hat{u}_k = 1$ and $\hat{t}_n = 1$, the play of G^2_1 is

$$\begin{array}{lcccccccc} \mathrm{I} & U_0 & & x(0),\lambda^0_0 & & U_1 & & x(2),\lambda^0_2 & \\ \mathrm{II} & & \langle 0,u_0\rangle & & x(1),\lambda^0_1 & & \langle 0,u_1\rangle & & x(3),\lambda^0_3 \cdots \end{array}$$

$$\begin{array}{cccccccccccc} \cdots U_{k-1} & & x(2k-2),\lambda^0_{2k-2} & & U_k & & T_0 & & x(2k),\lambda_0 & & T_1 & & x(2k+2),\lambda_2 & \\ & \langle 0,u_{k-1}\rangle & & x(2k-1),\lambda^0_{2k-1} & & \langle 1,-\rangle & & \langle 0,t_0\rangle & & x(2k+1),\lambda_1 & & \langle 0,t_1\rangle & & x(2k+3),\lambda_3 \cdots \end{array}$$

$$\begin{array}{cccccccccc} \cdots T_{n-1} & & x(2k+2n-2),\lambda_{2n-2} & & T_n & & x(2k+2n),\xi_0 & & x(2k+2n+2),\xi_2 & \\ & \langle 0,t_{n-1}\rangle & & x(2k+2n-1),\lambda_{2n-1} & & \langle 1,-\rangle & & x(2k+2n+1),\xi_1 & & x(2k+2n+3),\xi_3 \cdots. \end{array}$$

Any λ^0_i's played must be properly ordered with respect to $\langle D_\alpha | \alpha < \omega \cdot m\rangle$ using $\langle \omega^{L(\#^1_3(0))}_{i+1} | i < m\rangle$. Otherwise, the moves of G^2_1 must satisfy all of the conditions required of the moves for G^2_0. The sequence

$$\langle (U_k; \langle \hat{u}_k, u_k\rangle) | \forall j < k\ \hat{u}_j = 0\rangle \text{ [respectively, } \langle (T_n; \langle \hat{t}_n, t_n\rangle) | \forall j < n\ \hat{t}_j = 0\rangle]$$

and the Π^0_1 set B_1 [respectively, B_2] are related via R_1 [respectively, R_2]. Player I may only play $U_j \in L[\#^1_2]$. If $\hat{u}_k = 1$, player I may only play $T_n \in L(U_k)[\#^1_1]$ and the λ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (k+1)\rangle$ using $\langle \omega^{L(\#^1_2(U_k))}_{i+1} | i \le k\rangle$. If $\hat{u}_k = 1$ and $\hat{t}_n = 1$, the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1)\rangle$ using $\langle \omega^{L(\#^1_1(U_k, T_n))}_{i+1} | i \le n\rangle$.

Player I wins G^2_1 iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. Since G^2_1 is an open game, define for each ordinal α, P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in ON} P_\alpha$. If p is a legal position in G_1^2, let ℓ_p denote the set of legal positions in G_1^2 consistent with p. The set ℓ of legal positions for G_1^2 is in $L[\#_2^1]$.

Use $\langle P_\alpha | \alpha \in ON \rangle$ and Theorem 0.14 to define a wellordering $\prec$ of ℓ and the canonical w.s. s_1^2 for G_1^2 so that Lemma 2.1.1 (below) and the following hold: s_1^2 is definable in $L[\#_2^1]$, and if p is a legal position in G_1^2, then $s_1^2 | \ell_p$ is a w.s. for $(G_1^2)_p$ and is definable in any inner model of ZF in which $\prec | \ell_p$ is definable.

Lemma 2.1.1. Let p be a legal position in G_1^2. Then $s_1^2 | \ell_p$ is a w.s. for $(G_1^2)_p$ and each of the following hold:

iii.) If p includes the move $\langle \hat{u}_k, u_k \rangle$ and $\hat{u}_k = 1$, then $s_1^2 | \ell_p$ is definable in $L(U_k)[\#_2^1]$ from $\langle \omega_{i+1}^{L(\#_2^1(U_k))} | i \leq k \rangle$.

iv.) If p includes moves $\langle \hat{u}_k, u_k \rangle$ and $\langle \hat{t}_n, t_n \rangle$ such that $\hat{u}_k = \hat{t}_n = 1$, then $s_1^2 | \ell_p$ is definable in $L(U_k, T_n)$ from $\langle \omega_{i+1}^{L(\#_1^1(U_k, T_n))} | i \leq n \rangle$.

By Properties (iii) and (iv) and the proof of Theorem 1.1, we have the following:

Lemma 2.1.2. If

$$p_k = (U_0; \langle 0, u_0 \rangle; x(0), \lambda_0^0; x(1), \lambda_1^0; U_1; \langle 0, u_1 \rangle; x(2), \lambda_2^0; x(3), \lambda_3^0; ...$$

$$...; U_{k-1}; \langle 0, u_{k-1} \rangle; x(2k-2), \lambda_{2k-2}^0; x(2k-1), \lambda_{2k-1}^0; U_k; \langle 1, - \rangle)$$

is a legal position of G_1^2 and $A' = A(U_k; u_0, u_1, ..., u_{k-1}, \bar{x}(2k))$, then the following hold:

v.) The w.s. $s_1^2|\ell_{p_k}$ can be integrated so as to obtain a w.s. $s' \in L(\#_2^1(U_k))$ for A' and s' is a w.s. of the player for whom s_1^2 is a w.s.

vi.) If s_1^2 is a w.s. for I, p is a position consistent with s', and the moves in p of player II are consistent with $\bar{x}(2k)$ and $\langle u_i | i < k\rangle$, then $p \in U_k$. Therefore, if s_1^2 is a w.s. for I and x is a play consistent with s', then $x \in A(u_0, u_1, u_2, ..., u_{k-1}, \bar{x}(2k))$.

vii.) If $p \in U_k$ is a position consistent with s' and the moves in p of the player for whom s' is not a w.s. are compatible with $\langle u_i | i < k\rangle$, then p is consistent with each u_i.

Now we integrate s_1^2 similarly to the manner in which s_0^2 is integrated in Theorem 2.0, except that s_1^2 is integrated with respect to λ_i^0's using the uncountable set C_1^2 of indiscernibles for $L[\#_2^1]$.

Claim I: Player I has a w.s. for G_A if he has one for G_1^2.

Now assume $\langle\ \rangle \in P$ so that $s_1^2 \in L[\#_2^1]$ is a w.s. for I in G_1^2. We use s_1^2 to define a w.s. s for I in $\mathrm{G_A}$. Let

$$U_0 = s_1^2(\langle\ \rangle),\ \langle \hat{u}_0, u_0\rangle = \langle 1, -\rangle, \text{ and } p_0 = (U_0; \langle 1, -\rangle).$$

By Lemma 2.1.2(v), obtain a w.s. s_0 for $A(U_0)$ by integrating the w.s. $s_1^2|\ell_{p_0}$ for $(G_1^2)_{p_0}$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), 0)$. If $R_1(\bar{x}(i), 0)$ holds at every position, then $x \in A(U_0)$ so that $x \in A$ by Lemma 2.1.2(vi).

Suppose we reach a position such that $\neg R_1(\bar{x}(i_0), 0)$ and $\forall j < i_0 R_1(\bar{x}(j), 0)$.

We have defined

$$s(x(1); x(3); ...; x(2j-1)) = x(2j) \text{ for } 2j < i_0. \qquad \text{(viii)}$$

Since i_0 is odd by (ii), let $\langle \hat{u}_0, u_0\rangle = \langle 0, \bar{x}(i_0)\rangle$ and $p'_0 = (U_0; \langle 0, u_0\rangle)$. Define $s(\ )$ to be $x(0)$, where $(x(0), \lambda^0_0) = s^2_1(p'_0)$. Choose $\lambda^0_1 \in C^2_1$ and

$$p_1 = p'_0 * (x(0), \lambda^0_0; x(1), \lambda^0_1; U_1; \langle 1, -\rangle)$$

so that p_1 is consistent with s^2_1. By Lemma 2.1.2(v), obtain a w.s. s_1 for $A(U_1; u_0, \bar{x}(2))$ by integrating the w.s. $s^2_1|\ell_{p_1}$ for $(G^2_1)_{p_1}$, and let $s(p) = s_1(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), 1)$. By (vii), this definition of s is consistent with (viii). If $R_1(\bar{x}(i), 1)$ holds at every position, then $x \in A(U_1; u_0, \bar{x}(2))$ so that $x \in A$ by Lemma 2.1.2. If we reach a position such that $\neg R_1(\bar{x}(i_1), 1)$ and $\forall j < i_1 R_1(\bar{x}(j), 1)$, then continue to define s by integrating s^2_1 in the same manner as above.

In general, suppose we reach a position such that

$$\neg R_1(\bar{x}(i_j), j) \text{ and } \forall i < i_j R_1(\bar{x}(i), j)$$

for $j = 1, 2, 3, ..., k-1$. Choose $\lambda^0_1, \lambda^0_3, ..., \lambda^0_{2k-1} \in C^2_1$ and

$$p_k = (U_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda^0_0; x(1), \lambda^0_1; U_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda^0_2; x(3), \lambda^0_3; ...$$
$$...; U_{k-1}; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2), \lambda^0_{2k-2}; x(2k-1), \lambda^0_{2k-1}; U_k; \langle 1, -\rangle)$$

so that p_k is consistent with s^2_1. By Lemma 2.1.2(v), obtain a w.s. $s_k \in L(\#^1_2(U_k))$ for

$$A(U_k; u_0, u_1, ...u_{k-1}, \bar{x}(2k))$$

by integrating the w.s. $s^2_1|\ell_{p_k}$ for $(G^2_1)_{p_k}$, and let $s(p) = s_k(p)$ for any position

$p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), k)$.

Claim: The strategy s of player I is a w.s. for G_A.

Let x be a play of G_A consistent with s. First assume there is a least k such that $B_1(x, k)$. Then for each $j < k$, there is a least i_j such that $\neg R_1(\bar{x}(i_j), j)$. By the definition of s, there exist

$$U_0, U_1, U_2, ..., U_k \text{ and } \lambda_1^0, \lambda_3^0, \lambda_5^0, ..., \lambda_{2k-1}^0 \in C_1^2$$

such that the position

$$p_k = (U_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda_0^0; x(1), \lambda_1^0; U_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda_2^0; x(3), \lambda_3^0; ...$$
$$...; U_{k-1}; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2), \lambda_{2k-2}^0; x(2k-1), \lambda_{2k-1}^0; U_k; \langle 1, -\rangle)$$

is consistent with s_1^2. We obtained the w.s. s_k for

$$A(U_k; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k))$$

by integrating the w.s. $s_1^2|\ell_{p_k}$ for $(G_1^2)_{p_k}$ and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), k)$. Since $B_1(x, k)$, $\forall j R_1(\bar{x}(j), k)$. Therefore, x is a play consistent with s_k so that

$$x \in A(U_k; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k)).$$

Clearly, x is consistent with each $\bar{x}(i_j)$ and by Lemma 2.1.2(vi), $\forall j\ \bar{x}(j) \in U_k$. Thus, $x \in A$ if $\exists k B_1(x, k)$.

Now assume $\neg B_1(x, k)$ for all k. Then for each k, there is a least i_k such that $\neg R_1(\bar{x}(i_k), k)$. By the definition of s, for each n there exist a sequence $\vec{\lambda}_n = \langle \lambda_{2j+1}^0 | j < n\rangle$ of elements from C_1^2 and a position

$$p_n' = (U_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda_0^0; x(1), \lambda_1^0; U_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda_2^0; x(3), \lambda_3^0;$$

$U_2; \langle 0, \bar{x}(i_2)\rangle; x(4), \lambda_4^0; x(5), \lambda_5^0; ...; x(2n-2), \lambda_{2n-2}^0; x(2n-1), \lambda_{2n-1}^0; U_n; \langle 0, \bar{x}(i_n)\rangle\rangle$

which is consistent with s_1^2. ($\vec{\lambda}_j$ is not necessarily a subsequence of $\vec{\lambda}_{j+1}$.) By the following lemma, show $x \in D^*_{\omega\cdot m} = D$ by using the elements of C_1^2 to integrate s_1^2 with respect to the λ_{2i+1}^0's:

Lemma 2.1.3. Let

$$p = (U_0; \langle 0, u_0\rangle; x(0), \lambda_0^0; x(1), \lambda_1^0; U_1; \langle 0, u_1\rangle; x(2), \lambda_2^0; x(3), \lambda_3^0; ...$$
$$...; U_{n-1}; \langle 0, u_{n-1}\rangle; x(2n-2), \lambda_{2n-2}^0; x(2n-1), \lambda_{2n-1}^0; U_n)$$

and

$$p' = (U_0'; \langle 0, u_0'\rangle; x'(0), \tilde{\lambda}_0^0; x'(1), \tilde{\lambda}_1^0; U_1'; \langle 0, u_1'\rangle; x'(2), \tilde{\lambda}_2^0; x'(3), \tilde{\lambda}_3^0; ...$$
$$...; U_{n-1}'; \langle 0, u_{n-1}'\rangle; x'(2n-2), \tilde{\lambda}_{2n-2}^0; x'(2n-1), \tilde{\lambda}_{2n-1}^0; U_n')$$

be legal positions of G_1^2 consistent with s_1^2. If λ_{2i-1}^0 and $\tilde{\lambda}_{2i-1}^0$ are elements of C_1^2 for $i \le n$ and both the Borel auxiliary moves and the integer moves of player II are the same for p and p' (i.e. $x(2i-1) = x'(2i-1)$ and $u_i = u_i'$), then both the Borel auxiliary moves and integer moves of player I are the same for both p and p' (i.e. $x(2i) = x'(2i)$ and $U_i = U_i'$).

Since $x \in D$, $\forall k \neg B_1(x, k)$, and $B_2 \subseteq B_1$, $x \in A$. Thus, in either case, x is a win for I, and s is a w.s. of I.

Claim II: Player II has a w.s. for G_A if he has one for G_1^2.

Again, we use the uncountable set C_1^2 of indiscernibles for $L[\#_2^1]$ to integrate s_1^2 with respect to the λ_i^0's and otherwise we integrate s_1^2 similarly to the manner in which s_0^2 is integrated in Theorem 2.0.

Assume $\langle\ \rangle \notin$ P. Let

$$U_0 = \{\text{positions } u \text{ in } G_A | \forall U' \in L[\#_2^1]\ \langle 0, u\rangle \neq s_1^2(U')\}.$$

Then $\forall u \in U_0 \forall U' \in L[\#_2^1]\ \langle 0, u\rangle \neq s_1^2(U')$. (ix)

x.) If $(U'; \langle 0, u\rangle)$ is a legal position of G_1^2, then for each ordinal α, $P_\alpha \cap \ell_{(U';\langle 0,u\rangle)}$ is definable in $L[\#_2^1]$ from $\langle \omega_{i+1}^{L(\#_3^1(0))} | i < m\rangle$.

Therefore, $U_0 \in L[\#_2^1]$. Also, by (ix), $\langle 1, -\rangle = s_1^2(U_0)$. Let $p_0 = (U_0; \langle 1, -\rangle)$. By Lemma 2.1.2(v), obtain a w.s. s_0 for $A(U_0)$ by integrating the w.s. $s_1^2|\ell_{p_0}$ for $(G_1^2)_{p_0}$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i\ \bar{x}(j) \in U_0$. If $\bar{x}(i) \in U_0$ holds at every position, then $x \notin A(U_0)$ so that $x \notin A$.

Suppose we reach a position such that $\bar{x}(i_0) \notin U_0$ and $\forall j < i_0\ \bar{x}(j) \in U_0$. We have defined

$$s(x(0); x(2); ...; x(2j)) = x(2j+1) \text{ for } 2j < i_0. \qquad \text{(xi)}$$

Since $\bar{x}(i_0) \notin U_0$, there exists $U_0' \in L[\#_2^1]$ such that $\langle 0, \bar{x}(i_0)\rangle = s_1^2(U_0')$.

Let $\langle \hat{u}_0, u_0\rangle = \langle 0, \bar{x}(i_0)\rangle$ and let $p_1' = (U_0'; \langle 0, u_0\rangle; x(0), \lambda_0^0)$ be a position consistent with s_1^2 in which $\lambda_0^0 \in C_2^1$. Define $x(1) = s(x(0))$ to be $s_1^2(p_1')$. Let $U_1 = \{\text{positions } u \text{ in } G_A(\bar{x}(2), u_0) | \forall U' \in L[\#_2^1]\ \langle 0, u\rangle \neq s_1^2(U_0'; x(0), \lambda_1^0; U')\}$. Then $U_1 \in L[\#_2^1]$ and $\langle 1, -\rangle = s_1^2(U_0'; x(0), \lambda_0^0; U_1)$. Let

$$p_1 = p_0' * (x(1), \lambda_1^0; U_1; \langle 1, -\rangle).$$

By Lemma 2.1.2(v), obtain a w.s. s_1 for $A(U_1; u_0, \bar{x}(2))$ by integrating the w.s. $s_1^2|\ell_{p_1}$ for $(G_1^2)_{p_1}$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); ...; x(i-$

1)) such that $\forall j \leq i R_1(\bar{x}(j), 1)$. By Lemma 2.1.2(vi), this definition of s is consistent with (xi). If $\bar{x}(i) \in U_1$ holds at every position, then $x \notin A(U_1; u_0, \bar{x}(2))$ so that $x \notin A$. If we reach a position such that $\bar{x}(i_1) \notin U_1$, then continue to define s by integrating s_1^2 in the same manner as above.

In general, suppose we reach a position such that

$$\bar{x}(i_j) \notin U_j \text{ and } \forall i < i_j \; \bar{x}(i) \in U_j \text{ for } j = 1, 2, 3, ..., k-1.$$

Then let

$$x(0), x(2), x(4), ..., x(2k-2); \; U_0, U_1, U_2, ..., U_{k-1};$$

and $\lambda_0^0, \lambda_2^0, ..., \lambda_{2k-2}^0 \in C_1^2$ be such that the position

$$p'_{k-1} = (U_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda_0^0; x(1), \lambda_1^0; U_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda_2^0; x(3), \lambda_3^0; ...$$
$$...; U_{k-1}; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2), \lambda_{2k-2}^0)$$

is consistent with s_1^2. Define $s(x(0); x(2); ...; x(2k-2)) = x(2k-1)$ to be such that $p'_{k-1} * (x(2k-1), \lambda_{2k-1}^0)$ is consistent with s_1^2 for some λ_{2k-1}^0. Let

$$U_k = \{\text{positions } u \text{ in } G_A(\bar{x}(i_0), \bar{x}(i_1), \bar{x}(i_2), ..., \bar{x}(i_{k-1}), \bar{x}(2k)) |$$
$$\forall U' \in L[\#_2^1] \; \langle 0, u\rangle \neq s_1^2(p'_{k-1} * (x(2k-1), \lambda_{2k-1}^0; U'))\}.$$

Then $U_k \in L[\#_2^1]$ and $\langle 1, -\rangle = s_1^2(p'_{k-1} * (x(2k-1), \lambda_{2k-1}^0; U_k))$. Let

$$p_k = p'_{k-1} * (x(2k-1), \lambda_{2k-1}^0; U_k; \langle 1, -\rangle).$$

By Lemma 2.1.2(v), obtain a w.s. s_k for $A(U_k; u_0, u_1, ...u_{k-1}, \bar{x}(2k))$ by integrating the w.s. $s_1^2|\ell_{p_k}$ for $(G_1^2)_{p_k}$, and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i \; \bar{x}(j) \in U_k$.

Claim: The strategy s of player II is a w.s. for G_A.

Let x be a play of G_A consistent with s. First consider the case in which there is a least k such that $\forall i\ \bar{x}(i) \in U_k$. Then for each $j < k$, there is a least i_j such that $\bar{x}(i_j) \notin U_j$. By the definition of $x(2j+1) = s(x(0); x(2); ...; x(2j))$ for $j < k$ and by the definition of the U_i's, there exist

$$U'_0, U'_1, U'_2, ..., U'_{k-1} \in L[\#^1_2];\ \lambda^0_0, \lambda^0_2, \lambda^0_4, ..., \lambda^0_{2k-2} \in C^2_1; \text{ and}$$

$$p_k = (U'_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda^0_0; x(1), \lambda^0_1; U'_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda^0_2; x(3), \lambda^0_3; ...$$
$$...; U'_{k-1}; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2), \lambda^0_{2k-2}; x(2k-1), \lambda^0_{2k-1}; U_k; \langle 1, -\rangle)$$

such that p_k is consistent with s^2_1. We obtained the w.s. s_k for

$$A(U_k; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k))$$

by integrating the w.s. $s^2_1|\ell_{p_k}$ for $(G^2_1)_{p_k}$ and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i\ \bar{x}(j) \in U_k$. Since we are assuming $\forall j\ \bar{x}(j) \in U_k$, x is a play consistent with s_k so that $x \notin A(U_k; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k))$. Therefore, $x \notin A$.

Now assume for each k, there is a least i_k such that $\bar{x}(i_k) \notin U_k$. By the definition of $x(2j+1) = s(x(0); x(2); ...; x(2j))$ for $j < k$ and by the definition of the U_i's, there exists for each j, $U'_j \in L[\#^1_2]$ such that the following holds: For each n, there exist a sequence $\vec{\lambda}_n = \langle \lambda^0_{2j} | j < n\rangle$ of elements from C^2_1 and a position

$$p'_{n-1} = (U'_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda^0_0; x(1), \lambda^0_1; U'_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda^0_2; x(3), \lambda^0_3; ...$$
$$...; U'_{n-1}; \langle 0, \bar{x}(i_{n-1})\rangle; x(2n-2), \lambda^0_{2n-2})$$

such that the position p'_{n-1} is consistent with s^2_1. ($\vec{\lambda}_k$ is not necessarily a subsequence of $\vec{\lambda}_{k+1}$.) Since each position p'_k is consistent with s^2_1, $\neg R_1(\bar{x}(i_k), k)$ so that $\forall k \neg B_1(x,k)$. By the following lemma, show $x \notin D$ by using the elements of C^2_1 to integrate s^2_1 with respect to the λ^0_{2i}'s:

Lemma 2.1.4. Let

$$p = (U_0; \langle 0, u_0\rangle; x(0), \lambda^0_0; x(1), \lambda^0_1; U_1; \langle 0, u_1\rangle; x(2), \lambda^0_2; x(3), \lambda^0_3; \ldots$$
$$\ldots; U_{n-1}; \langle 0, u_{n-1}\rangle; x(2n-2), \lambda^0_{2n-2}; x(2n-1), \lambda^0_{2n-1})$$

and

$$p' = (U'_0; \langle 0, u'_0\rangle; x'(0), \tilde{\lambda}^0_0; x'(1), \tilde{\lambda}^0_1; U'_1; \langle 0, u'_1\rangle; x'(2), \tilde{\lambda}^0_2; x'(3), \tilde{\lambda}^0_3; \ldots$$
$$\ldots; U'_{n-1}; \langle 0, u'_{n-1}\rangle; x'(2n-2), \tilde{\lambda}^0_{2n-2}; x'(2n-1), \tilde{\lambda}^0_{2n-1})$$

be legal positions of G^2_1 consistent with s^2_1. If λ^0_{2i} and $\tilde{\lambda}^0_{2i}$ are elements of C^2_1 for $i < n$ and both the Borel auxiliary moves and the integer moves of player I are the same for p and p' (i.e. $x(2i) = x'(2i)$ and $U_i = U'_i$), then both the Borel auxiliary moves and integer moves of player II are the same for both p and p' (i.e. $x(2i-1) = x'(2i-1)$ and $u_i = u'_i$).

Since $\forall k \neg B_1(x,k)$ and $x \notin D$, x is a win for II. Consequently, s is a w.s. in G_A of the player for whom s^2_1 is w.s. ∎

Now we generalize G^2_1:

Definition 2.1. Let $B_2, B_1 \in \Pi^0_1$, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $k \in \omega$ strongly witness $A \in (2 * \Pi^0_1)^*_+$. Then we refer to the auxiliary game G^2_1 described in the Proof of Theorem 2.1 as *the* G^2_1 *auxiliary game determined by* $B_2, B_1, \in \Pi^0_1$, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, *and* $k \in \omega$ if $m = k+1$ and $D_\alpha = A_\alpha$ for $\alpha < \omega \cdot m$.

Suppose $\vec{Q} = \langle Q_i | i < \beta \rangle$ and $\vec{q} = \langle q_i | i < \gamma \rangle$ respectively are a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of G_A. Then *the* $G^2_1(\vec{Q}; \vec{q})$ *auxiliary game determined by*

$$B_2, B_1, \ \langle A_\alpha \mid \alpha < \omega^2 \rangle, \ \textit{and } m \in \omega$$

is the game in which player I wins iff a position is reached at which II cannot make a (legal) move, which has exactly the same moves as G^2_1, and these moves are subject to the following conditions:

i.) The sequence $\langle (U_n; \langle \hat{u}_n, u_n \rangle) | \forall j < n \ \hat{u}_j = 0 \rangle$ of Borel auxiliary moves and the Π^0_1 set B_1 are related via R_{B_1}. The sequence $\langle (T_n; \langle \hat{t}_n, t_n \rangle) | \forall j < n \ \hat{t}_j = 0 \rangle$ of Borel auxiliary moves and the Π^0_1 set B_2 are related via R_{B_2}. (R_{B_i} is defined as in Theorem 2.1.)

ii.) Each $\bar{x}(i) \in \bigcap_{j<\beta} Q_j$ and each $\bar{x}(i)$ must be consistent with every q_j.

iii.) $U_i \in L[\#^1_2]$, and if $\hat{u}_k = 1$, then $T_i \in L(U_k)[\#^1_1]$.

iv.) The λ^0_i's are properly ordered with repect to $\langle A_\alpha | \alpha < \omega \cdot (m+1) \rangle$ using $\langle \omega^{L(\#^1_3(\vec{Q}))}_{i+1} | i \leq m \rangle$. If $\hat{u}_k = 1$, then the λ_i's are properly ordered with repect to $\langle A_\alpha | \alpha < \omega \cdot (k+1) \rangle$ using $\langle \omega^{L(\#^1_2(\vec{Q}, U_k))}_{i+1} | i \leq k \rangle$.

v.) If $\hat{u}_k = \hat{t}_n = 1$, the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1) \rangle$ using $\langle \omega^{L(\#^1_1(\vec{Q}, U_k))}_{i+1} | i \leq n \rangle$.

These conditions are analogous to the conditions for the moves of G^2_1. The first is a condition which the moves of G^2_1 also must satisfy. The others are derived by changing the conditions for the moves of G^2_1 so that we obtain conditions which are consistent with $\vec{Q}$ and $\vec{q}$. We refer to $G_1^2(\vec{Q};\vec{q})$ instead of the $G_1^2(\vec{Q};\vec{q})$ auxiliary game determined by B_2, B_1, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $m \in \omega$ whenever B_2, B_1, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $m \in \omega$ are clear from the context. Analogous to Theorem 2.1, we have the following:

Corollary 2.1.1. Let B_2, B_1, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, m, A, $\vec{Q}$, and $\vec{q}$ be as in Definition 2.1. Let p be a legal position of a game G^* such that the moves of G^* following p constitute a play of $G_1^2(\vec{Q};\vec{q})$. Suppose $\vec{Q}$ has a wellordering which is definable in $L(\vec{Q})$, $\#_3^1(\vec{Q})$ exists, and s^* is a w.s. for G^* such that $s^*|\ell_p \in L(\vec{Q})[\#_2^1]$. Then $s^*|\ell_p$ can be integrated so as to obtain a w.s. $s_p \in L(\#_3^1(\vec{Q}))$ for $A(\vec{Q};\vec{q})$ such that the following hold:

i.) s_p is a w.s. of the player for whom s^* is a w.s.,

ii.) If s^* is a w.s. for I, $\hat{p}$ is a position consistent with s_p, and the moves in $\hat{p}$ of player II are consistent with $\vec{q}$, then $\hat{p} \in \bigcap_{i<\beta} Q_i$. Therefore, if s^* is a w.s. for I and x is a play consistent with s_p, then $x \in A(\vec{q})$.

iii.) Let $\hat{p}$ be a position consistent with s_p and with $\vec{Q}$. If the moves in $\hat{p}$ of the player for whom s_p is not a w.s. are consistent with $\vec{q}$, then $\hat{p}$ is consistent with $\vec{q}$. ∎