

§2. The Existence of $\#_{\beta}^{\gamma}(0)$ Implies Determinacy.

In this section, we generalize the results of Section One. We show the following two results:

Theorem 2.4. If $L(\beta\#_{\gamma+1}^1(0))[\#_{\gamma}^1] \models$ “ $r\#_{\gamma}^1$ exists for every real r ,” then $\text{Det}(\gamma * \Pi_1^0, \beta * \Sigma_1^0)^*$.

Theorem 2.5. If $0^{(\beta+1)}\#_{\gamma+1}^1$ exists, then $\text{Det}(\gamma * \Pi_1^0, \beta * \Sigma_1^0)_+^*$.

The proof of Theorems 2.4 and 2.5 is by induction on γ and β . We also prove by induction two slightly stronger results: Corollaries 2.4.1 and 2.5.1. The statement of these corollaries is very similar to the statements of Theorems 2.4 and 2.5.

We first prove Theorems 2.4 and 2.5 and Corollaries 2.4.1 and 2.5.1 for $\beta = 0$ and any γ . The proof is by induction on γ . However, we need as our induction hypothesis, Corollaries 2.4.1 and 2.5.1 with $\beta = 0$ and γ replaced by $\gamma - 1$. Theorems 2.4 and 2.5 with $\beta = 0$ and γ replaced by $\gamma - 1$ are not strong enough to be our induction hypothesis. Once we have shown Theorems 2.4 and 2.5 and Corollaries 2.4.1 and 2.5.1 for $\beta = 0$, we prove the four general results by induction on β .

Theorems 1.0 and 1.1 (of Section One) respectively are Corollaries 2.4.1 and 2.5.1 with $\beta = 0$ and $\gamma = 1$. Corollaries 2.2.1 and 2.3.1 respectively are Corollaries 2.4.1 and 2.5.1 with $\beta = 0$. We could use Theorems 1.0 and 1.1 as the Base Step for our inductive proof, with respect to γ , of Corollaries 2.2.1

and 2.3.1. However, we first show Theorems 2.4 and 2.5 and Corollaries 2.4.1 and 2.5.1 for the case in which $\gamma = 2$ and $\beta = 0$. Figure 6 illustrates the Theorems of this section.

Recall that $\#_2^1$ is the partial sharp function whose domain is a subset of the reals and which sends r to $r^{\#_2^1}$. Now we characterize

$$L[\#_2^1] \models \text{“}r^{\#_2^1} \text{ exists for every real } r\text{”}$$

in terms of determinacy.

Theorem 2.0. If $L[\#_2^1] \models \text{“}r^{\#_2^1} \text{ exists for every real } r\text{”}$, then $\text{Det}(\Pi_1^0, \Pi_1^0)^*$.

Proof: Assume $L[\#_2^1] \models \text{“}r^{\#_2^1} \text{ exists for every real } r\text{”}$. Let $B_2, B_1 \in \Pi_1^0$ and $\langle A_\alpha \mid \alpha < \omega^2 \rangle$ strongly witness $A \in (\Pi_1^0, \Pi_1^0)^*$. Wlog $B_2 \subseteq B_1$. Also, there exist R_1 and R_2 in Δ_1^0 such that

$$\text{i.) } B_i(x, n) \leftrightarrow \forall k R_i(\bar{x}(k), n),$$

and

$$\text{ii.) if } \neg R_i(\bar{x}(k), n) \text{ and } \forall j < k R_i(\bar{x}(j), n), \text{ then } k \text{ is odd.}$$

We show that G_A has a w.s. s . Condition (ii) helps to simplify the proof.

We describe an open game G_0^2 which has a w.s. $s_0^2 \in L[\#_2^1]$. We integrate s_0^2 to get the w.s. $s \in L[\#_2^1]$ for G_A . The moves of G_0^2 are the integer moves $x(i)$ of G_A as well as two types of auxiliary moves: Borel auxiliary moves and ordinal auxiliary moves.

G_0^2 contains a sequence

$$\langle (U_k; \langle \hat{u}_k, u_k \rangle) \mid k \in \omega \ \& \ \forall j < k \ \hat{u}_j = 0 \rangle$$

of Borel auxiliary moves, which is related to the Π_1^0 set B_1 via R_1 . Player I may only play $U_k \in L[\#_2^1]$. If II plays $\hat{u}_k = 0$ for all $k \in \omega$, then the play of G_0^2 is

$$\begin{array}{ccccccc} \text{I} & U_0 & x(0) & U_1 & x(2) & \cdots & U_{n-1} \\ \text{II} & \langle 0, u_0 \rangle & x(1) & \langle 0, u_1 \rangle & x(3) & \cdots & \langle 0, u_{n-1} \rangle \cdots \end{array}$$

If II plays $\hat{u}_k = 1$, then the moves of G_0^2 following $\langle \hat{u}_k, u_k \rangle = \langle 1, - \rangle$ constitute a play of the auxiliary game $G_1^1(U_k; u_0, u_1, u_2, \dots, u_{k-1}, \bar{x}(2k))$ determined by B_2 , $\langle A_\alpha | \alpha < \omega \rangle$, and $k \in \omega$ (with integer moves $x(2k+i)$ for $i \in \omega$). Suppose $\hat{u}_k = 1$. Then for $j > k$, no Borel auxiliary moves $(U_j; \langle \hat{u}_j, u_j \rangle)$ are played. Following the move $\langle \hat{u}_k, u_k \rangle = \langle 1, - \rangle$ are the integer moves $x(2k+i)$ for $i \in \omega$ and a sequence $\langle (T_n; \langle \hat{t}_n, t_n \rangle) | \forall j < n \hat{t}_j = 0 \rangle$ of Borel auxiliary moves which is related to B_2 via R_2 . Player I may only play $T_n \in L(U_k)[\#_1^1]$. Also, if $\forall j < i \hat{t}_j = 0$, then ordinal auxiliary moves λ_{2i} and λ_{2i+1} are respectively played with the integer moves $x(2k+2i)$ and $x(2k+2i+1)$ so that the λ_i 's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (k+1) \rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(U_k))} | i \leq k \rangle$. If II plays $\hat{t}_j = 0$ for all $j \in \omega$, then the play of G_0^2 is

$$\begin{array}{ccccccc} \text{I} & U_0 & x(0) & U_1 & x(2) & \cdots & \\ \text{II} & \langle 0, u_0 \rangle & x(1) & \langle 0, u_1 \rangle & x(3) & \cdots & \\ & \cdots & U_{k-1} & x(2k-2) & U_k & \langle 1, - \rangle & T_0 & x(2k), \lambda_0 & T_1 & x(2k+2), \lambda_2 & \cdots \\ & & \langle 0, u_{k-1} \rangle & x(2k-1) & & & \langle 0, t_0 \rangle & x(2k+1), \lambda_1 & \langle 0, t_1 \rangle & x(2k+3), \lambda_3 & \cdots \end{array}$$

If II plays $\hat{t}_n = 1$, then each move of G_0^2 following $\langle \hat{t}_n, t_n \rangle$ consists of an integer $x(2k+2n+i)$ and an ordinal ξ_i , and the ξ_i 's must be properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1) \rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(U_k, T_n))} | i \leq n \rangle$. If $\hat{t}_n = 1$, the play of G_0^2 is

$$\begin{array}{l}
\text{I} \quad U_0 \quad x(0) \quad U_1 \quad x(2) \quad \cdots \\
\text{II} \quad \langle 0, u_0 \rangle \quad x(1) \quad \langle 0, u_1 \rangle \quad x(3) \quad \cdots \\
\cdots \quad U_{k-1} \quad x(2k-2) \quad U_k \quad T_0 \quad x(2k), \lambda_0 \quad T_1 \quad x(2k+2), \lambda_2 \quad \cdots \\
\quad \langle 0, u_{k-1} \rangle \quad x(2k-1) \quad \langle 1, - \rangle \quad \langle 0, t_0 \rangle \quad x(2k+1), \lambda_1 \quad \langle 0, t_1 \rangle \quad x(2k+3), \lambda_3 \quad \cdots \\
\cdots \quad T_{n-1} \quad x(2k+2n-1), \lambda_{2n-1} \quad T_n \quad x(2k+2n), \xi_0 \quad x(2k+2n+2), \xi_2 \quad \cdots \\
\quad \langle 0, t_{n-1} \rangle \quad x(2k+2n-2), \lambda_{2n-2} \quad \langle 1, - \rangle \quad x(2k+2n+1), \xi_1 \quad x(2k+2n+3), \xi_3 \quad \cdots
\end{array}$$

Player I wins G_0^2 iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. G_0^2 is an open game and therefore we define, for each ordinal α , P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in ON} P_\alpha$. If p is a legal position in G_0^2 , let ℓ_p denote the set of legal positions in G_0^2 consistent with p . The set ℓ of legal positions for G_0^2 is in $L[\#_2^1]$.

Use $\langle P_\alpha | \alpha \in ON \rangle$ and Theorem 0.14 to define a wellordering \prec of ℓ and the canonical w.s. s_0^2 for G_0^2 so that Lemma 2.0.1 (below) and the following hold: s_0^2 is definable in $L[\#_2^1]$, and if p is a legal position in G_0^2 , then $s_0^2|_{\ell_p}$ is a w.s. for $(G_0^2)_p$ and is definable in any inner model of ZF in which $\prec|_{\ell_p}$ is definable.

Lemma 2.0.1. Let p be a legal position in G_0^2 . Then $s_0^2|_{\ell_p}$ is a w.s. for $(G_0^2)_p$ and each of the following hold:

iii.) If p includes the move $\langle \hat{u}_k, u_k \rangle$ and $\hat{u}_k = 1$, then $s_0^2|_{\ell_p}$ is definable in $L(U_k)[\#_1^1]$ from $\langle \omega_{i+1}^{L(\#_2^1(U_k))} | i \leq k \rangle$.

iv.) If p includes moves $\langle \hat{u}_k, u_k \rangle$ and $\langle \hat{t}_n, t_n \rangle$ such that $\hat{u}_k = \hat{t}_n = 1$, then $s_0^2|_{\ell_p}$ is definable in $L(U_k, T_n)$ from $\langle \omega_{i+1}^{L(\#_1^1(U_k, T_n))} | i \leq n \rangle$.

By Theorem 1.1 and Lemma 2.0.1, we have the following:

Lemma 2.0.2. If

$$p_k = (U_0; \langle 0, u_0 \rangle; x(0); x(1); U_1; \langle 0, u_1 \rangle; x(2); x(3); \dots \\ \dots; U_{k-1}; \langle 0, u_{k-1} \rangle; x(2k-2); x(2k-1)); U_k; \langle 1, - \rangle)$$

is a legal position of G_0^2 and $A' = A(U_k; u_0, u_1, \dots, u_{k-1}, \bar{x}(2k))$, then the following hold:

v.) The w.s. $s_0^2|_{\ell_{p_k}}$ can be integrated so as to obtain a w.s. $s' \in L(\#_2^1(U_k))$ for A' and s' is a w.s. of the player for whom s_0^2 is a w.s.

vi.) If s_0^2 is a w.s. for I, p is a position consistent with s' , and the moves in p of player II are consistent with $\bar{x}(2k)$ and $\langle u_j | j < k \rangle$, then $p \in U_k$. Hence, if s_0^2 is a w.s. for I and x is a play consistent with s' , then $x \in A(u_0, u_1, u_2, \dots, u_{k-1}, \bar{x}(2k))$.

vii.) If $p \in U_k$ is a position consistent with s' and the moves in p of the player for whom s' is not a w.s. are compatible with $\langle u_j | j < k \rangle$, then p and each u_j are compatible.

Now we define a w.s. for G_A :

Claim I: Player I has a w.s. for G_A if he has one for G_0^2 .

Let's first consider the case in which $\langle \rangle \in P$. Then $s_0^2 \in L[\#_2^1]$ is a w.s. for I in G_0^2 . We use s_0^2 to define a w.s. s for I in G_A . Let

$$U_0 = s_0^2(\langle \rangle), \langle \hat{u}_0, u_0 \rangle = \langle 1, - \rangle, \text{ and } p_0 = (U_0; \langle 1, - \rangle).$$

By Lemma 2.0.2(v), obtain a w.s. $s_0 \in L(\#_2^1(U_0))$ for $A(U_0)$ by integrating the w.s. $s_0^2|_{\ell_{p_0}}$ for $(G_0^2)_{p_0}$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); \dots; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), 0)$. If $R_1(\bar{x}(i), 0)$ holds at

every position, then $x \in A(U_0)$ so that $x \in A$ by Lemma 2.0.2(vi).

Suppose we reach a position such that $\neg R_1(\bar{x}(i_0), 0)$ and $\forall j < i_0 R_1(\bar{x}(j), 0)$.

We have defined

$$s(x(1); x(3); \dots; x(2j - 1)) = x(2j) \text{ for } 2j < i_0. \quad (\text{viii})$$

Since i_0 is odd by (ii), let $\langle \hat{u}_0, u_0 \rangle = \langle 0, \bar{x}(i_0) \rangle$ and $p'_0 = (U_0; \langle 0, u_0 \rangle)$. Define $x(0) = s(\)$ to be $s_0^2(p'_0)$ and U_1 to be $s_0^2(p'_0 * (x(0); x(1)))$. Let

$$p_1 = p'_0 * (x(0); x(1); U_1; \langle 1, - \rangle).$$

By Lemma 2.0.2(v), obtain a w.s. $s_1 \in L(\#_2^1(U_1))$ for $A(U_1; u_0, \bar{x}(2))$ by integrating the w.s. $s_0^2|_{\ell_{p_1}}$ for $(G_0^2)_{p_1}$, and let $s(p) = s_1(p)$ for any position $p = (x(0); x(1); \dots; x(i - 1))$ such that $\forall j \leq i R_1(\bar{x}(j), 1)$. By (vii), this definition of s is consistent with (viii). If $R_1(\bar{x}(i), 1)$ holds at every position, then $x \in A(U_1; u_0, \bar{x}(2))$ so that $x \in A$ by Lemma 2.0.2. If we reach a position such that $\neg R_1(\bar{x}(i_1), 1)$ and $\forall j < i_1 R_1(\bar{x}(j), 1)$, then continue to define s by integrating s_0^2 in the same manner as above.

In general, suppose we reach a position such that

$$\neg R_1(\bar{x}(i_j), j) \text{ and } \forall i < i_j R_1(\bar{x}(i), j)$$

for $j = 1, 2, 3, \dots, k-1$. Then let $x(0), x(2), x(4), \dots, x(2k)$ and $U_0, U_1, U_2, \dots, U_k$ be such that the position

$$p_k = (U_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); U_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots \\ \dots; U_{k-1}; \langle 0, \bar{x}(i_{k-1}) \rangle; x(2k - 2); x(2k - 1); U_k; \langle 1, - \rangle)$$

is consistent with s_0^2 . By Lemma 2.0.2(v), obtain a w.s. $s_k \in L(\#_2^1(U_k))$ for

$$A(U_k; u_0, u_1, \dots, u_{k-1}, \bar{x}(2k))$$

by integrating the w.s. $s_0^2|_{\ell_{p_k}}$ for $(G_0^2)_{p_k}$, and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); \dots; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), k)$.

Claim: The strategy s of player I is a w.s. for G_A .

Let x be a play of G_A consistent with s . First consider the case in which there is a least k such that $B_1(x, k)$. Then for each $j < k$, there is a least i_j such that $\neg R_1(\bar{x}(i_j), j)$. Furthermore, there exist $U_0, U_1, U_2, \dots, U_k$ such that the position

$$\begin{aligned} p_k = & (U_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); U_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots \\ & \dots; U_{k-1}; \langle 0, \bar{x}(i_{k-1}) \rangle; x(2k-2); x(2k-1); U_k; \langle 1, - \rangle) \end{aligned}$$

is consistent with s_0^2 . By the definition of s , we obtained the w.s. s_k for

$$A(U_k; \bar{x}(i_0), \bar{x}(i_1), \dots, \bar{x}(i_{k-1}), \bar{x}(2k))$$

by integrating the w.s. $s_0^2|_{\ell_{p_k}}$ for $(G_0^2)_{p_k}$ and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); \dots; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), k)$. Since $B_1(x, k)$, $\forall j R_1(\bar{x}(j), k)$. Therefore, x is a play consistent with s_k so that

$$x \in A(U_k; \bar{x}(i_0), \bar{x}(i_1), \dots, \bar{x}(i_{k-1}), \bar{x}(2k)).$$

Clearly, x is consistent with each $\bar{x}(i_j)$ and by Lemma 2.0.2(vi), $\forall j \bar{x}(j) \in U_k$.

Thus, $x \in A$ if $\exists k B_1(x, k)$.

Now assume $\neg B_1(x, k)$ for all k . Then for each k , there is a least i_k such that $\neg R_1(\bar{x}(i_k), k)$. Furthermore, there exists $\langle U_j | j \in \omega \rangle$ such that for each k , the position

$$p_k = (U_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); U_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots \\ \dots; U_{k-1}; \langle 0, \bar{x}(i_{k-1}) \rangle; x(2k-2); x(2k-1); U_k; \langle 1, - \rangle)$$

is consistent with s_0^2 . Since each p_k is consistent with s_0^2 , the play

$$y = (U_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); U_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); U_2; \langle 0, \bar{x}(i_2) \rangle; x(4); x(5); \dots \\ \dots; U_n; \langle 0, \bar{x}(i_n) \rangle; x(2n); x(2n+1); \dots)$$

is consistent with s_0^2 . Since y is a play of G_0^2 in which player II made only legal moves, y is a win for II, contradicting s_0^2 being a w.s. for I. Thus, $\exists k B_1(x, k)$, $x \in A$, and s is a w.s. for player I.

Claim II: Player II has a w.s. for G_A if he has one for G_0^2 .

Now let's consider the case $\langle \rangle \notin P$. We integrate II's w.s. $s_0^2 \in L[\#_2^1]$ for G_0^2 to get the w.s. s for II in G_A . Let

$$U_0 = \{\text{positions } u \text{ in } G_A \mid \forall U' \in L[\#_2^1] \langle 0, u \rangle \neq s_0^2(U')\}.$$

Then $\forall u \in U_0 \forall U' \in L[\#_2^1] \langle 0, u \rangle \neq s_0^2(U')$. (ix)

x.) If $(U'; \langle 0, u \rangle)$ is a legal position of G_0^2 , then for each ordinal α , $P_\alpha \cap \ell_{(U'; \langle 0, u \rangle)}$ is definable in $L[\#_2^1]$.

Therefore, $U_0 \in L[\#_2^1]$. Also, by (ix), $\langle 1, - \rangle = s_0^2(U_0)$. Let $p_0 = (U_0; \langle 1, - \rangle)$.

By Lemma 2.0.2(v), obtain a w.s. s_0 for $A(U_0)$ by integrating the w.s. $s_0^2 \upharpoonright \ell_{p_0}$ for $(G_0^2)_{p_0}$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); \dots; x(i-1))$ such that $\forall j \leq i \bar{x}(j) \in U_0$. If $\bar{x}(i) \in U_0$ holds at every position, then $x \notin A(U_0)$ so that $x \notin A$.

Suppose we reach a position such that $\bar{x}(i_0) \notin U_0$ and $\forall j < i_0 \bar{x}(j) \in U_0$.

We have defined

$$s(x(0); x(2); \dots; x(2j)) = x(2j + 1) \text{ for } 2j < i_0. \quad (\text{xi})$$

Since $\bar{x}(i_0) \notin U_0$, there exists $U'_0 \in L[\#_2^1]$ such that $\langle 0, \bar{x}(i_0) \rangle = s_0^2(U'_0)$.

Let $\langle \hat{u}_0, u_0 \rangle = \langle 0, \bar{x}(i_0) \rangle$ and $p'_1 = (U'_0; \langle 0, u_0 \rangle; x(0))$. Define $x(1) = s(x(0))$ to be $s_0^2(p'_1)$. Let

$$U_1 = \{\text{positions } u \text{ in } G_A(\bar{x}(2), u_0) \mid \forall U' \in L[\#_2^1] \langle 0, u \rangle \neq s_0^2(U'_0; x(0); U')\}.$$

Then $U_1 \in L[\#_2^1]$ and $\langle 1, - \rangle = s_0^2(U'_0; x(0); U_1)$. Let

$$p_1 = p'_0 * (x(1); U_1; \langle 1, - \rangle).$$

By Lemma 2.0.2(v), obtain a w.s. $s_1 \in L(\#_2^1(U_1))$ for $A(U_1; u_0, \bar{x}(2))$ by integrating the w.s $s_0^2|_{\ell_{p_1}}$ for $(G_0^2)_{p_1}$, and let $s(p) = S_1(p)$ for any position $p = (x(0); x(1); \dots; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), 1)$. By Lemma 2.0.2(vii), this definition of s is consistent with (xi). If $\bar{x}(i) \in U_1$ holds at every position, then $x \notin A(U_1; u_0, \bar{x}(2))$ so that $x \notin A$. If we reach a position such that $\bar{x}(i_1) \notin U_1$, then continue to define s by integrating s_0^2 in the same manner as above.

In general, suppose we reach a position such that

$$\bar{x}(i_j) \notin U_j \text{ and } \forall i < i_j \bar{x}(i) \in U_j \text{ for } j = 1, 2, 3, \dots, k-1.$$

Then let $x(0), x(2), x(4), \dots, x(2k-2)$ and $U'_0, U'_1, U'_2, \dots, U'_{k-1}$ be such that the position

$$\begin{aligned} p'_{k-1} = & (U'_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); U'_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots \\ & \dots; U'_{k-1}; \langle 0, \bar{x}(i_{k-1}) \rangle; x(2k-2)) \end{aligned}$$

is consistent with s_0^2 . Let $x(2k-1) = s_0^2(p'_{k-1})$ and

$$U_k = \{ \text{positions } u \text{ in } G_A \text{ consistent with } \bar{x}(2k) \text{ and } \langle \bar{x}(i_j) | j < k \rangle | \\ \forall U' \in L[\#_2^1] \langle 0, u \rangle \neq s_0^2(p'_{k-1} * (x(2k-1); U')) \}.$$

Then $p_k = p'_{k-1} * (x(2k-1); U_k; \langle 1, - \rangle)$ is consistent with s_0^2 . By Lemma 2.0.2(v), obtain a w.s. s_k for

$$A(U_k; u_0, u_1, \dots, u_{k-1}, \bar{x}(2k))$$

by integrating the w.s. $s_0^2|_{\ell_{p_k}}$ for $(G_0^2)_{p_k}$, and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); \dots; x(i-1))$ such that $\forall j \leq i \bar{x}(j) \in U_k$.

Claim: The strategy s of player II is a w.s. for G_A .

Let x be a play of G_A consistent with s . First consider the case in which there is a least k such that $\forall i \bar{x}(i) \in U_k$. Then for each $j < k$, there is a least i_j such that $\bar{x}(i_j) \notin U_j$. By the definition of $x(2j+1) = s(x(0); x(2); \dots; x(2j))$ for $j < k$ and by the definition of the U_i 's, there exist $U'_0, U'_1, U'_2, \dots, U'_{k-1}$ such that the position

$$p_k = (U'_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); U'_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots \\ \dots; U'_{k-1}; \langle 0, \bar{x}(i_{k-1}) \rangle; x(2k-2); x(2k-1); U_k; \langle 1, - \rangle)$$

is consistent with s_0^2 . We obtained the w.s. s_k for

$$A(U_k; \bar{x}(i_0), \bar{x}(i_1), \dots, \bar{x}(i_{k-1}), \bar{x}(2k))$$

by integrating the w.s. $s_0^2|_{\ell_{p_k}}$ for $(G_0^2)_{p_k}$ and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); \dots; x(i-1))$ such that $\forall j \leq i \bar{x}(j) \in U_k$. Since we are assuming $\forall j \bar{x}(j) \in U_k$, x is a play consistent with s_k so that

$$x \notin A(U_k; \bar{x}(i_0), \bar{x}(i_1), \dots, \bar{x}(i_{k-1}), \bar{x}(2k)).$$

Therefore, $x \notin A$.

Now assume for each k , there is a least i_k such that $\bar{x}(i_k) \notin U_k$. By the definition of $x(2j+1) = s(x(0); x(2); \dots; x(2j))$ for $j < k$ and by the definition of the U_i 's, there exist $U'_0, U'_1, U'_2, \dots, U'_k$ such that the position

$$(U'_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); U'_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots; U'_k; \langle 0, \bar{x}(i_k) \rangle)$$

is consistent with s_0^2 . Since each such position is consistent with $s_0^2, \neg R_1(\bar{x}(i_k), k)$.

Since $B_2 \subseteq B_1, \forall j \in \{1, 2\} \forall k \neg B_j(x, k)$ so that x is a win for II. Consequently, s is a w.s. in G_A of the player for whom s_0^2 is w.s. ■

We now generalize the result of Theorem 2.0, but first we define an auxiliary game $G_0^2(\vec{U}; \vec{u})$ which is a generalization of the auxiliary game G_0^2 of Theorem 2.0.

Definition 2.0. Let $B_2, B_1 \in \Pi_1^0$ and $\langle A_\alpha \mid \alpha < \omega^2 \rangle$ strongly witness $A \in (2 * \Pi_1^0)^*$. Then we refer to the auxiliary game G_0^2 described in the Proof of Theorem 2.0 as

the G_0^2 auxiliary game determined by $B_2, B_1, \in \Pi_1^0$, and $\langle A_\alpha \mid \alpha < \omega^2 \rangle$.

Suppose $\vec{Q} = \langle Q_i \mid i < \beta \rangle$ and $\vec{q} = \langle q_i \mid i < \gamma \rangle$ respectively are a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of G_A . Then *the $G_0^2(\vec{Q}; \vec{q})$ auxiliary game determined by*

$$B_2, B_1, \text{ and } \langle A_\alpha \mid \alpha < \omega^2 \rangle$$

is the game in which player I wins iff a position is reached at which II cannot

make a (legal) move, which has exactly the same moves as G_0^2 , and these moves are subject to the following conditions:

i.) The sequence $\langle (U_n; \langle \hat{u}_n, u_n \rangle) | \forall j < n \hat{u}_j = 0 \rangle$ of Borel auxiliary moves and the Π_1^0 set B_1 are related via R_{B_1} . The sequence

$$\langle (T_n; \langle \hat{t}_n, t_n \rangle) | \forall j < n \hat{t}_j = 0 \rangle$$

of Borel auxiliary moves and the Π_1^0 set B_2 are related via R_{B_2} .

ii.) Each $\bar{x}(i) \in \bigcap_{j < \beta} Q_j$ and each $\bar{x}(i)$ must be consistent with every q_j .

iii.) $U_i \in L[\#_2^1]$, and if $\hat{u}_k = 1$, then $T_i \in L(U_k)[\#_1^1]$.

iv.) If $\hat{u}_k = 1$, then the λ_i 's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (k + 1) \rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}, U_k))} | i \leq k \rangle$.

v.) If $\hat{u}_k = \hat{t}_n = 1$, the ξ_i 's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n + 1) \rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, U_k, T_n))} | i \leq n \rangle$.

These conditions are analogous to the conditions for the moves of G_0^2 . The first is a condition which the moves of G_0^2 also must satisfy. The others are derived by changing the conditions for the moves of G_0^2 so that we obtain conditions which are consistent with \vec{Q} and \vec{q} . We refer to $G_0^2(\vec{Q}; \vec{q})$ instead of the $G_0^2(\vec{Q}; \vec{q})$ auxiliary game determined by B_2, B_1 , and $\langle A_\alpha | \alpha < \omega^2 \rangle$ whenever B_2, B_1 , and $\langle A_\alpha | \alpha < \omega^2 \rangle$ are clear from the context. Analogous to Theorem 2.0, we have the following:

Corollary 2.0.1. Let $B_2, B_1, \langle A_\alpha | \alpha < \omega^2 \rangle, A, \vec{Q}$, and \vec{q} be as in Definition 2.0. Let \hat{p} be a legal position of a game G^* such that the moves of G^*

following \hat{p} constitute a play of $G_0^2(\vec{Q}; \vec{q})$. Suppose \vec{Q} has a wellordering which is definable in $L(\vec{Q})$, $L(\vec{Q})[\#_2^1] \models "r^{\#_2^1} \text{ exists for every real } r,"$ and s^* is a w.s. for G^* such that $s^*|_{\ell_{\hat{p}}} \in L(\vec{Q})[\#_2^1]$. Then $s^*|_{\ell_{\hat{p}}}$ can be integrated so as to obtain a w.s. $s_{\hat{p}} \in L(\vec{Q})[\#_2^1]$ for $A(\vec{Q}; \vec{q})$ such that the following hold:

- i.) $s_{\hat{p}}$ is a w.s. of the player for whom s^* is a w.s.,
- ii.) If s^* is a w.s. for I, p is a position consistent with $s_{\hat{p}}$, and the moves of player II in p are consistent with \vec{q} , then $p \in \bigcap_{i < \beta} Q_i$. Therefore, if s^* is a w.s. for I and x is a play consistent with $s_{\hat{p}}$, then $x \in A(\vec{q})$.
- iii.) Let p be a position consistent with both $s_{\hat{p}}$ and \vec{Q} . If the moves in p of the player for whom $s_{\hat{p}}$ is not a w.s. are consistent with \vec{q} , then p is consistent with \vec{q} . ■